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PERTURBED ROTATIONAL MOTIONS OF A RIGID BODY SIMILAR TO REGULAR PRECESSION*

D.D. LESHCHENKO and S.N. SALLAM

Perturbed rotational motions of a rigid body, similar to regular precession in the Lagrange case, when the restoring torque depends on the angle of nutation, are investigated. It is assumed that the angular velocity of the body is fairly large, its direction are close to the dynamic axis of symmetry of the body and the perturbing torques are small compared with the restoring torques. A small parameter is introduced in a special way and the method of averaging is used. The averaged equations of motion are derived in the first and second approximations. Specific mechanical models of the perturbations are considered.

Lagrange-like perturbed motions of a rigid body were also investigated in /1/. Almost regular perturbed rotational motions of a rigid body have been studied /2, 3/,**(**See also LESHCHENKO D.D. and SALLAM S.N., Perturbed rotational motions of a rigid body with mass distribution close to the Lagrange case. Odessa, 1988. Dep. at UkrNIINTI 28.06.1988, No.1656 - Uk 88.) and some attention has been given to pseudoregular***(***LESHCHENKO D.D. and SALLAM S.N., Perturbed motions of a rigid body similar to pseudoregular precession. Odessa, 1988. Dep. at UkrNIINTI 28.06.1988, No.1656 - Uk 88.) precession in the Lagrange case; in the former case it was assumed that a constant restoring torque is applied to the body.

1. Statement of the problem. We consider the motion of a dynamically symmetric rigid body about a fixed point O due to a perturbing torque and a restoring torque depending on the angle of nutation θ.

The equations of motion (dynamic and kinematic Euler equations) have the form

$$
\begin{gathered}
Ap^{\cdot} + (C - A)qr = k(\theta)\sin\theta\cos\varphi + M_1 \\
Aq^{\cdot} + (A - C)pr = -k(\theta)\sin\theta\sin\varphi + M_2 \\
Cr^{\cdot} = M_3,\ M_i = M_i(p, q, r, \psi, \theta, \varphi, t)\ (i = 1, 2, 3) \\
\psi^{\cdot} = (p\sin\varphi + q\cos\varphi)/\sin\theta,\ \theta^{\cdot} = p\cos\varphi - q\sin\varphi \\
\varphi^{\cdot} = r - (p\sin\varphi + q\cos\varphi)\operatorname{ctg}\theta
\end{gathered}
\tag{1.1}
$$

The dynamic equations are written in terms of projections on the body's principal axes of inertia, which pass through the point O. Thus p, q, r are the projections of the angular velocity vector on these axes, M_i $(i = 1, 2, 3)$ are the projections of the vector of the perturbing torque on the same axes, assumed to be periodic functions of the Euler angles ψ, θ, φ with periods 2π, and A is the equatorial and C is the axial moment of inertia about the point O, $A \neq C$.

It is assumed that a restoring torque $k(\theta)$ depending on the angle of nutation is applied to the body. In the case of a heavy top we have $k = mgl$, where m is the mass of the body, g is the acceleration due to gravity and l is the distance from the fixed point O to the centre of gravity of the body.

The perturbing torques M_i in (1.1) are assumed to be known functions of their arguments. If there are no perturbations $(M_i = 0,\ i = 1, 2, 3)$ and $k(\theta) = \text{const}$, Eqs.(1.1) are those of the Lagrange case.

We will make the following assumptions:

$$
p^2 + q^2 \ll r^2,\ Cr^2 \gg k,\ |M_i| \ll k\ (i = 1, 2, 3)
\tag{1.2}
$$

which mean that the direction of the angular velocity of the body is close to the dynamic axis of symmetry, the angular velocity is large enough to impart to the body kinetic energy

*Prikl.Matem.Mekhan.,54,2,224-232,1990

significantly exceeding the potential energy due to the restoring torque, and the perturbing torques are small compared with the restoring torques. Inequalities (1.2) justify the introduction of a small parameter $\varepsilon \ll 1$, so that

$$p = \varepsilon P,\ q = \varepsilon Q,\ k(\theta) = \varepsilon K(\theta),\ M_i = \varepsilon^2 M_i^*(P, Q, r, \psi, \theta, \varphi, t) \quad (i = 1, 2, 3) \tag{1.3}$$

Conditions (1.2) and (1.3) were also adopted in /3/, but there the restoring torque k was assumed to be constant. In /2/ the third inequality of (1.2) was replaced by the condition $|M_i| \ll k\ (i = 1, 2),\ M_3 \sim k$.

The new variables P, Q and the variables and constants $r, \psi, \theta, \varphi, K, A, C, M_i^*$ are assumed to be bounded of the order of unity as $\varepsilon \to 0$. Our problem is to investigate the asymptotic behaviour of system (1.1) for small ε, provided that conditions (1.2) and (1.3) hold. We shall use the method of averaging /4-6/ over a time interval of length $\sim\varepsilon^{-1}$.

2. ***Averaging procedure.*** Let us apply the transformation of variables (1.3) to system (1.1). Dividing both sides of the first two equations by ε, we obtain

$$\begin{gathered}
AP^{\cdot} + (C - A)Qr = K(\theta)\sin\theta\cos\varphi + \varepsilon M_1^* \\
AQ^{\cdot} + (A - C)Pr = -K(\theta)\sin\theta\sin\varphi + \varepsilon M_2^* \\
Cr^{\cdot} = \varepsilon^2 M_3^*,\ \psi^{\cdot} = \varepsilon(P\sin\varphi + Q\cos\varphi)/\sin\theta \\
\theta^{\cdot} = \varepsilon(P\cos\varphi - Q\sin\varphi),\ \varphi^{\cdot} = r - \varepsilon(P\sin\varphi + Q\cos\varphi)\operatorname{ctg}\theta
\end{gathered} \tag{2.1}$$

Considering the zeroth-approximation system, we put $\varepsilon = 0$ in (2.1). Then the last four equations of (2.1) yield

$$r = r_0,\ \psi = \psi_0,\ \theta = \theta_0,\ \varphi = r_0 t + \varphi_0 \tag{2.2}$$

Here $r_0, \psi_0, \theta_0, \varphi_0$ are constants - the initial values of the appropriate variables at $t = 0$. Substituting expressions (2.2) into the first two equations of system (2.1) with $\varepsilon = 0$ and integrating the resulting system of linear equations for P, Q, we obtain

$$\begin{gathered}
P = a\cos\gamma_0 + b\sin\gamma_0 + \lambda_0\sin(r_0 t + \varphi_0) \\
Q = a\sin\gamma_0 - b\cos\gamma_0 + \lambda_0\cos(r_0 t + \varphi_0) \\
a = P_0 - \lambda_0\sin\varphi_0,\ b = -Q_0 + \lambda_0\cos\varphi_0 \\
\lambda_0 = K_0 C^{-1} r_0^{-1}\sin\theta_0,\ \gamma_0 = n_0 t,\ n_0 = (C - A)A^{-1} r_0 \neq 0 \\
|n_0/r_0| \leqslant 1,\quad K_0 = K(\theta_0)
\end{gathered} \tag{2.3}$$

Here P_0, Q_0 are the initial values of the variables P, Q defined in (1.3), and the variable $\gamma = \gamma_0$ has the meaning of the phase of the oscillations. System (2.1) is essentially non-linear, and therefore we introduce an additional variable γ, defined by the equation

$$\gamma^{\cdot} = n,\ \gamma(0) = 0,\ n = (C - A)A^{-1} r \tag{2.4}$$

For $\varepsilon = 0$ we have $\gamma = \gamma_0 = n_0 t$ by (2.3) Eqs.(2.2) and (2.3) determine the general solution of system (2.1), (2.4) when $\varepsilon = 0$.

Using (2.2), we eliminate the constants from the first two equations of (2.3) and solve the resulting equations for a and b:

$$\begin{gathered}
a = P\cos\gamma + Q\sin\gamma - \lambda\sin(\gamma + \varphi) \\
b = P\sin\gamma - Q\cos\gamma + \lambda\cos(\gamma + \varphi);\ \lambda = KC^{-1}r^{-1}\sin\theta
\end{gathered} \tag{2.5}$$

Define a new variable δ as follows:

$$r = r_0 + \varepsilon\delta \tag{2.6}$$

We now consider system (2.1) for $\varepsilon \neq 0$ and Eqs.(2.5) and (2.6) as transformation formulae (involving the variable γ) from variables P, Q, r to variables a, b, δ. Using these formulae, we transform system (2.1) and (2.4) from variables $P, Q, r, \psi, \theta, \varphi, \gamma$ to new variables $a, b, \delta, \psi, \theta, \alpha, \gamma$, where

$$\alpha = \gamma + \varphi \tag{2.7}$$

After some reduction we obtain the following system:

$$\begin{aligned}
a^{\cdot} = {} & \varepsilon A^{-1}(M_1^\circ\cos\gamma + M_2^\circ\sin\gamma) - \varepsilon KD_{11}\cos\theta\,(b - KD_{11}\sin\theta\cos\alpha) \\
& - \varepsilon K'D_{11}\sin\theta\sin\alpha\,(a\cos\alpha + b\sin\alpha) + \varepsilon^2 KD_{12}\,\delta\cos\theta\,(b -
\end{aligned} \tag{2.8}$$

$$2KD_{11}\sin\theta\cos\alpha) + \varepsilon^2 K'D_{12}\delta\sin\theta\sin\alpha\,(a\cos\alpha +$$
$$b\sin\alpha) + \varepsilon^2 KD_{22}M_3^\circ\sin\theta\sin\alpha$$
$$b^{\cdot} = \varepsilon A^{-1}(M_1^\circ\sin\gamma - M_2^\circ\cos\gamma) + \varepsilon KD_{11}\cos\theta\,(a + KD_{11}\sin\theta\sin\alpha)$$
$$\varepsilon K'D_{11}\sin\theta\cos\alpha\,(a\cos\alpha + b\sin\alpha) - \varepsilon^2 KD_{12}\,\delta\cos\theta\,(a +$$
$$2KD_{11}\sin\theta\sin\alpha) - \varepsilon^2 K'D_{12}\,\delta\sin\theta\cos\alpha\,(a\cos\alpha + b\sin\alpha) -$$
$$\varepsilon^2 KD_{22}M_3^0\sin\theta\cos\alpha$$
$$\delta^{\cdot} = \varepsilon C^{-1}M_3^\circ,\ \psi^{\cdot} = \varepsilon\,(a\sin\alpha - b\cos\alpha)/\sin\theta + \varepsilon KD_{11} - \varepsilon^2 KD_{12}\delta$$
$$\theta^{\cdot} = \varepsilon\,(a\cos\alpha + b\sin\alpha),\ \gamma^{\cdot} = n_0 + \varepsilon\,(C - A)A^{-1}\delta$$
$$\alpha^{\cdot} = CA^{-1}r_0 + \varepsilon CA^{-1}\delta - \varepsilon\operatorname{ctg}\theta\,(a\sin\alpha - b\cos\alpha) - \varepsilon KD_{11}\cos\theta +$$
$$\varepsilon^2 KD_{12}\delta\cos\theta$$
$$D_{ij} = C^{-i}r_0^{-j},\ K' = dK/d\theta$$

Here M_i° are the functions obtained from M_i^* (see (1.3)) after the substitution (2.5)-(2.7):

$$M_i^\circ(a, b, \delta, \psi, \theta, \alpha, \gamma, t) = M_i^*(P, Q, r, \psi, \theta, \varphi, t)\ (i = 1, 2, 3) \tag{2.9}$$

The system of Eqs.(2.8) has the form

$$\begin{aligned} x^{\cdot} &= \varepsilon F_1(x, y) + \varepsilon^2 F_2(x, y),\ x(0) = x_0 \\ y^{1\cdot} &= \omega_1 + \varepsilon g_1(x, y) + \varepsilon^2 g_2(x, y),\ y^1(0) = y^{10} \\ y^{2\cdot} &= \omega_2 + \varepsilon h_1(x, y) + \varepsilon^2 h_2(x, y),\ y^2(0) = y^{20} \end{aligned} \tag{2.10}$$

The vector-valued function $x = (x^1, \ldots, x^5)$ consists of the slow variables $a, b, \delta, \psi, \theta$; the symbols y^1 and y^2 denote the fast variables α, γ; ω_1, ω_2 are constant phases, equal to $CA^{-1}r_0$ and $(C - A)A^{-1}r_0$, respectively. The vector-valued functions $F_i, g_i, h_i\ (i = 1, 2)$ are determined by the right-hand sides of Eqs.(2.8).

We denote the two-dimensional vector (g_1, h_1) by Z_1. We shall assume that the perturbing torques M_i^* do not depend on t.

Following the well-known procedure for constructing asymptotic formulae for system (2.10) /5/, we shall try to find a change of variables

$$\begin{aligned} x &= x^* + \varepsilon u_1(x^*, y^*) + \varepsilon^2 u_2(x^*, y^*) + \ldots \\ y &= y^* + \varepsilon v_1(x^*, y^*) + \varepsilon^2 v_2(x^*, y^*) + \ldots \\ y &= (y^1, y^2),\ x^* = (x^{*1}, \ldots, x^{*5}),\ y^* = (y^{*1}, y^{*2}) \end{aligned}$$

which reduces system (2.1) to the form

$$\begin{aligned} x^{*\cdot} &= \varepsilon A_1(x^*) + \varepsilon^2 A_2(x^*) + \ldots \\ y^{*\cdot} &= \omega + \varepsilon B_1(x^*) + \varepsilon^2 B_2(x^*) + \ldots,\ \omega = (\omega_1, \omega_2) \end{aligned} \tag{2.11}$$

To do this we must choose suitable functions u_1, u_2, v_1, v_2. The equations for the vector-valued functions u_1, v_1 are /5/

$$\begin{aligned} \omega\partial u_1/\partial y^* &= F_1(x^*, y^*) - A_1(x^*) \\ \omega\partial v_1/\partial y^* &= Z_1(x^*, y^*) - B_1(x^*) \end{aligned} \tag{2.12}$$

Here $(\partial f/\partial x)$ is the matrix of partial derivatives $\|\partial f_i/\partial x^j\|\ (i, j = 1, 2, \ldots, 5)$. The functions $A_1(x^*)$, $B_1(x^*)$ are defined by

$$\begin{aligned} A_1(x^*) &= \frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi} F_1(x^*, y^*)\,dy^{*1}\,dy^{*2} \\ B_1(x^*) &= \frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi} Z_1(x^*, y^*)\,dy^{*1}\,dy^{*2} \end{aligned} \tag{2.13}$$

The function $u_2(x^*, y^*)$ must be a solution of the equation

$$\begin{aligned} &\frac{\partial u_2}{\partial y^*}\,\omega = G(x^*, y^*) - A_2(x^*) \\ &G(x^*, y^*) = F_2(x^*, y^*) + \frac{\partial F_1}{\partial x^*}u_1 + \frac{\partial F_1}{\partial y^*}v_1 - \frac{\partial u_1}{\partial x^*}A_1(x^*) - \frac{\partial u_1}{\partial y^*}B_1(x^*) \end{aligned} \tag{2.14}$$

The function $A_2(x^*)$ is defined as follows:

$$A_2(x^*) = \frac{1}{4\pi^2} \int_0^{2\pi}\int_0^{2\pi} G(x^*, y^*)\, dy^{*1}\, dy^{*2} \tag{2.15}$$

We shall now determine the averaged system of equations to a first approximation for the slow variables

$$x_1^{*\cdot} = \varepsilon A_1(x_1^*), \quad x_1^*(0) = x_{10} \tag{2.16}$$

as well as the system to a second approximation for the slow variables

$$x_2^{*\cdot} = \varepsilon A_1(x_2^*) + \varepsilon^2 A_2(x_2^*), \; x_2^*(0) = x_0 \tag{2.17}$$

and the system of equations in the second approximation for the fast variables

$$y_2^{*\cdot} = \omega + \varepsilon B_1(x_1^*(t)), \; y_2^*(0) = y^\circ; \; y^\circ = (y^{10}, y^{20}) \tag{2.18}$$

which is readily integrated:

$$y_2^*(t) = y^\circ + \omega t + \varepsilon \int_0^t B_1(x_1^*(s))\, ds \tag{2.19}$$

To investigate the second-approximation system (2.17), we transform variables by putting $\tau = \varepsilon t$, giving system (2.17) the form

$$dx_2^*/d\tau = A_1(x_2^*) + \varepsilon A_2(x_2^*) \tag{2.20}$$

In this case the time interval $(0, T/\varepsilon)$ over which the solutions of the original system (2.10) are being investigated becomes an interval $(0, T)$ independent of ε. The solution of system (2.20) is assumed to have the form

$$x_2^*(\tau) = x^{(1)}(\tau) + \varepsilon x^{(2)}(\tau) + O(\varepsilon^2) \tag{2.21}$$

Substituting (2.21) into (2.20), we obtain the following systems of equations for the vector-valued functions $x^{(i)}(\tau) = x_i(t)$ $(\tau = \varepsilon t,\ i = 1, 2)$:

$$dx^{(1)}/d\tau = A_1(x^{(1)}), \quad x^{(1)}(0) = x_0, \tag{2.22}$$
$$dx^{(2)}/d\tau = A_1'(x^{(1)}(\tau))x^{(2)} + A_2(x^{(1)}(\tau)), \; x^{(2)}(0) = 0 \tag{2.23}$$

where A_1' is the matrix of partial derivatives of the vector-valued function $A_1(x)$: $A_1'(x) = \| \partial A_1^i/\partial x^j \|$. As system (2.22) is linear, it is often easier to investigate than system (2.20).
Let $X(\tau, c)$ denote the general solution of the first-approximation system (2.22):

$$X_\tau = A_1(X), \; X(0, c) = c = x_0 \tag{2.24}$$

Then the functions $x^{(1)}(\tau)$, $x^{(2)}(\tau)$ are given by the expressions

$$x^{(1)}(\tau) = X(\tau, x_0), \quad x^{(2)}(\tau) = \Phi(\tau) \int_0^\tau \Phi^{-1}(\tau_1)\, \eta(\tau_1)\, d\tau_1 \tag{2.25}$$

Here Φ is the fundamental matrix of the homogeneous equation corresponding to the second approximation:

$$\Phi(\tau) = \| \partial X(\tau, c)/\partial c \|_{c=x_0}, \; \eta(\tau) = A_2(x^{(1)}(\tau)) = A_2(X(\tau, x_0))$$

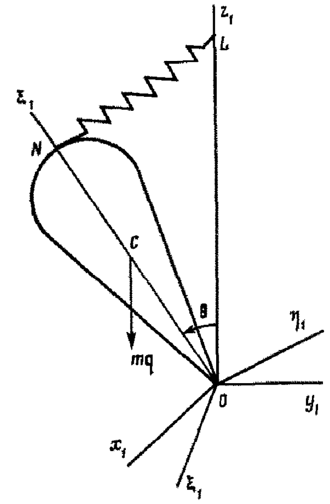
Define a vector-valued function

$$x_\varepsilon^v(t) = x^{(1)}(\varepsilon t) + \varepsilon x^{(2)}(\varepsilon t) + \varepsilon u_1(x^{(1)}(\varepsilon t), y^\circ + \omega t + \varepsilon \int_0^t B_1(x^{(1)}(\varepsilon s)\, ds) \tag{2.26}$$
$$y_\varepsilon^v(t) = y^\circ + \omega t + \varepsilon \int_0^t B_1(x^{(1)}(\varepsilon s))\, ds$$

The above formal procedure for construction the functions $x_\varepsilon^v(t)$, $y_\varepsilon^v(t)$ was justified in /8/.

Thus, the construction of approximate solutions $x_\varepsilon^v(t)$, $y_\varepsilon^v(t)$ reduces to the following procedure: use Fourier series to solve Eqs.(2.12) and (2.14); then use formula (2.15) to construct the vector-valued function $A_2(x^*)$; using (2.25), determine the solutions $x^{(1)}(\tau)$ and $x^{(2)}(\tau)$ of Eqs.(2.22) and (2.23); finally, use formula (2.26) to obtain the required approximations $x_\varepsilon^v(t)$, $y_\varepsilon^v(t)$. The procedure will now be implemented for a few specific systems of equations for the dynamics of a rigid body.

Our examples of perturbations will all be such that the Fourier expansions of the right-hand sides of Eqs.(2.12) and (2.14) contain only a finite number of terms. Hence the condition for Eqs.(2.12) and (2.14) to be solvable reduces to verification of a finite number of conditions of the form $\omega_1 m_1 + \omega_2 m_2 \neq 0$. In all our examples these conditions have the form $CA^{-1}r_0 \neq 0$, $(C-A)A^{-1}r_0 \neq 0$, and the latter are always satisfied thanks to our initial assumptions.



As an example of a restoring torque which depends on the angle of nutation, let us consider a rigid body with a spring attached to it at a point N, with the end L of the spring fixed (see the figure). The forces acting on the body are the force of gravity mg and the elastic force $\mathbf{F}$ of the spring, whose modulus is proportional to the deformation of the spring $F = \nu(s - s_0)$, where ν is the stiffness of the spring. In this case the restoring torque is

$$k(\theta) = mgl + \nu hz\,[1 - s_0(h^2 + z^2 - 2hz\cos\theta)^{-1/2}] \tag{2.27}$$

where $ON = z$, $OC = l$, $OL = h$, $LN = s = s(\theta)$.

By (1.3), $k(\theta) = \varepsilon K(\theta)$.

3. *The case of linear applied dissipative torques.*

We will now consider the perturbed motion of a rigid body in the Lagrange case, allowing for the torques applied to the body from the external medium. We shall assume that the perturbing torques M_i $(i = 1, 2, 3)$ (see (1.3)) have the form /7/

$$M_1 = -\varepsilon^2 I_1 P,\ M_2 = -\varepsilon^2 I_1 Q,\ M_3 = -\varepsilon^2 I_3 r,\ I_1, I_3 > 0 \tag{3.1}$$

where I_1 and I_3 are certain constants of proportionality which depend on the properties of the medium and the shape of the body.

The first three equations of (2.8) in this case, in variables $a, b, \delta, \psi, \theta, \alpha, \gamma$, become

$$\begin{gathered}
a^{\cdot} = -\varepsilon A^{-1}I_1(a + KD_{11}\sin\theta\sin\alpha) - \varepsilon KD_{11}\cos\theta\,(b - \\
KD_{11}\sin\theta\cos\alpha) + \varepsilon^2 A^{-1}I_1 KD_{12}\delta\sin\theta\sin\alpha + \varepsilon^2 KD_{12}\delta\cos\theta\,(b - \\
2KD_{11}\sin\theta\cos\alpha) - \varepsilon^2 I_3 KD_{21}\sin\theta\sin\alpha - \varepsilon K'D_{11}\sin\theta\sin\alpha\,(a\cos\alpha + \\
b\sin\alpha) + \varepsilon^2 K'D_{12}\delta\sin\theta\sin\alpha\,(a\cos\alpha + b\sin\alpha) \\
b^{\cdot} = -\varepsilon A^{-1}I_1(b - KD_{11}\sin\theta\cos\alpha) + \varepsilon KD_{11}\cos\theta\,(a + \\
KD_{11}\sin\theta\sin\alpha) - \varepsilon^2 KA^{-1}I_1 D_{12}\delta\sin\theta\cos\alpha - \varepsilon^2 KD_{12}\delta\cos\theta\,(a + \\
2KD_{11}\sin\theta\sin\alpha) + \varepsilon^2 I_3 KD_{21}\sin\theta\cos\alpha + \varepsilon K'D_{11}\sin\theta\cos\alpha\,(a\cos\alpha + \\
b\sin\alpha) - \varepsilon^2 K'D_{12}\delta\sin\theta\cos\alpha\,(a\cos\alpha + b\sin\alpha) \\
\delta^{\cdot} = -\varepsilon C^{-1}I_3 r_0 - \varepsilon^2 C^{-1}I_3\delta
\end{gathered} \tag{3.2}$$

The other equations of system (2.8) remain unchanged.

To construct an approximate solution of system (3.2), we will use the averaging procedure described in Sect.2. The vector-valued functions A_1 and B_1 are determined from formulae (2.13):

$$\begin{gathered}
A_1 = \{A_1^{(i)}\},\ (i = 1, 2, \ldots, 5),\ B_1 = \{B_1^{(j)}\},\ (j = 1, 2) \\
A_1^{(1)} = -A^{-1}I_1 a - KD_{11}b\cos\theta - {}^1/_2 K'D_{11}b\sin\theta \\
A_1^{(2)} = -A^{-1}I_1 b + KD_{11}a\cos\theta + {}^1/_2 K'D_{11}a\sin\theta \\
A_1^{(3)} = -C^{-1}I_3 r_0,\ A_1^{(4)} = KD_{11},\ A_1^{(5)} = 0 \\
B_1^{(1)} = CA^{-1}\delta - KD_{11}\cos\theta,\ B_1^{(2)} = (C - A)A^{-1}\delta
\end{gathered} \tag{3.3}$$

The fourth and fifth components of the vector-valued function $u_1\{u_1^{(i)}\}$ $(i = 1, 2, \ldots, 5)$ may be written

$$\begin{gathered}
u_1^{(4)} = -AD_{11}(a\cos\alpha + b\sin\alpha)/\sin\theta \\
u_1^{(5)} = AD_{11}(a\sin\alpha - b\cos\alpha)
\end{gathered} \tag{3.4}$$

Note that combinations of the type $M_1^\circ\cos\gamma + M_2^\circ\sin\gamma$ and $M_1^\circ\sin\gamma - M_2^\circ\cos\gamma$, as follows from Eqs.(2.8) and (3.2), do not depend on γ and the right-hand sides of these equations depend only on one fast variable α. This fact, pointed out in /3/, is analogous to the sufficient conditions obtained in /1/ for the averaging procedure to be applicable to the equations of motion with respect to the angle of nutation alone. The solution of Eqs.(2.12) is then simplified.

The vector-valued function $A_2(x^*)$, after suitable reduction using formula (2.15), may be written as

$$A_2(x^*) = \{A_2^{(i)}\} \quad (i = 1, 2, \ldots, 5) \tag{3.5}$$
$$A_2^{(1)} = KD_{12}\,[\delta b\cos\theta - {}^1/_2 KD_{21}Ab\,(3\cos^2\theta - 1) - I_1C^{-1}a\cos\theta] +$$
$${}^1/_2 K'D_{12}\delta b\sin\theta - {}^1/_8 AD_{33}b\,(K'\sin\theta)^2 - {}^1/_2 AK'D_{33}bK\sin 2\theta -$$
$${}^1/_2 I_1D_{22}ad\,(K\sin\theta)/d\theta$$
$$A_2^{(2)} = -KD_{12}\,[\delta a\cos\theta - {}^1/_2 KD_{21}Aa\,(3\cos^2\theta - 1) + I_1C^{-1}b\cos\theta] -$$
$${}^1/_2 K'D_{12}\delta a\sin\theta + {}^1/_8 AD_{33}a\,(K'\sin\theta)^2 + {}^1/_2 AK'D_{33}aK\sin 2\theta -$$
$${}^1/_2 I_1D_{22}bd\,(K\sin\theta)/d\theta$$
$$A_2^{(3)} = -C^{-1}I_3\delta, \quad A_2^{(4)} = KD_{12}\,(-\delta + KD_{21}\cos\theta)$$
$$A_2^{(5)} = I_1KD_{22}\sin\theta$$

Let us determine the solution of the averaged system of equations to a first approximation (2.16), taking into account (3.3), for the slow and fast variables:

$$a^{(1)} = \exp(-\varepsilon A^{-1}I_1t)\,(a^\circ\cos wt - b^\circ\sin wt) \tag{3.6}$$
$$b^{(1)} = \exp(-\varepsilon A^{-1}I_1t)\,(b^\circ\cos wt + a\sin wt)$$
$$\delta^{(1)} = -\varepsilon C^{-1}I_3r_0t, \quad \psi^{(1)} = \varepsilon K_0D_{11}t + \psi_0, \quad \theta^{(1)} = \theta_0$$
$$\alpha^{(1)} = CA^{-1}r_0t - \varepsilon K_0D_{11}\cos\theta_0 t - {}^1/_2\varepsilon^2A^{-1}I_3r_0t^2 + \varphi_0$$
$$\gamma^{(1)} = n_0t - {}^1/_2\varepsilon^2\,(C - A)A^{-1}C^{-1}I_3r_0t^2$$
$$w = {}^1/_2\varepsilon D_{11}\,(2K\cos\theta + K'\sin\theta)_{\theta=\theta_0}$$

where the quantities a°, b°, n_0 are determined by (2.3); ψ_0, θ_0 and φ_0 are constants, equal to the initial values of the Euler angles at $t = 0$. Comparison of expressions (3.6) for the slow variables $a^{(1)}, b^{(1)}$ with the parallel formulae (4.5) of /3/ in the case when $K = \mathrm{const}$ shows that the expressions are identical.

On the basis of these formulae, using (2.26), one can construct the components of the function $x_\varepsilon^v(t)$ corresponding to the variables ψ and θ, writing them as

$$\psi_\varepsilon^v(t) = \psi_0 + \varepsilon K_0D_{11}t + S^{(1)} \tag{3.7}$$
$$S^{(1)} = \varepsilon^2tK_0^2D_{33}\cos\theta_0 + {}^1/_2\varepsilon^4K_0D_{21}I_3t^2 -$$
$$\varepsilon AD_{11}\exp(-\varepsilon A^{-1}I_1t)\,C^\circ\sin(\alpha^{(1)} + \sigma)/\sin\theta_0$$
$$\theta_\varepsilon^v(t) = \theta_0 + \varepsilon^2tI_1K_0D_{22}\sin\theta_0 + \varepsilon AD_{11}\exp(-\varepsilon A^{-1}I_1t)\,C^\circ\sin(\alpha^{(1)} - \mu)$$
$$\cos\sigma = \sin\mu = b^{(1)}\exp(\varepsilon A^{-1}I_1t)/C^\circ, \quad C^\circ = (a^{\circ 2} + b^{\circ 2})^{1/2}$$

Comparison of these expressions with formulae (4.7) of /3/ shows that the two groups are identical at $K = K_0$. In formula (3.7) for θ_ε^v the term of order ε is the product of the slowly exponentially decreasing factor $\exp(-\varepsilon A^{-1}I_1t)$, representing energy dissipation, and the oscillating factor $\sin(\alpha^{(1)} - \mu)$.

The value of the damping constant and the nature of the slow variation of the phase of small oscillations for $b^{(1)}, \alpha^{(1)}$, can be read off from formulae (3.6), which differ from the parallel formulae (4.5) of /3/ in the value of w.

The term $S^{(1)}(\varepsilon, t)$ in the formula (3.7) for $\psi_\varepsilon^v(t)$ is of order ε over the time interval $(0, T\varepsilon^{-1})$. The expression for the angular velocity of precession $\omega_p = K_0C^{-1}r_0^{-1}$ is known from the approximate theory of gyroscopes /8/. Our expression for $S^{(1)}(\varepsilon, t)$ improves this formula for the problem.

For the example considered above, with the restoring torque given by formula (2.27) and taking formula (2.13) into consideration, the solution of the averaged system of equations in the first approximation (2.16) for $a^{(1)}, b^{(1)}, \delta^{(1)}, \theta^{(1)}, \gamma^{(1)}$ is of the form (3.6). Only the expressions for $\psi^{(1)}$ and $\alpha^{(1)}$ change; they may be written as

$$\psi^{(1)} = D_{11}k(\theta_0)\,t + \psi_0 \tag{3.8}$$
$$\alpha^{(1)} = CA^{-1}r_0t - D_{11}k(\theta_0)\,t\cos\theta_0 - {}^1/_2\varepsilon^2A^{-1}I_3r_0t^2 + \varphi_0$$

In (3.6),

$$w = D_{11}\,\{(mgl + \nu hz)\cos\theta_0 - {}^1/_2\nu hzs_0\,[2\,(h^2 + z^2)\cos\theta_0 -$$
$$5hz\cos^2\theta_0 + hz]\,(h^2 + z^2 - 2hz\cos\theta_0)^{-3/2}\}$$

while $k(\theta_0)$ in (3.8) is defined by (2.27) with $\theta = \theta_0$

The components of the function $x_\varepsilon^v(t)$ corresponding to the variables ψ and θ in our example have the form

$$\psi_\varepsilon^v(t) = \psi_0 + D_{11}k(\theta_0)t + S^{(1)} \tag{3.9}$$
$$S^{(1)} = D_{33}\cos\theta_0 k^2(\theta_0)t + {}^1/_2\varepsilon^2 D_{21} I_3 k(\theta_0)t^2 -$$
$$\varepsilon D_{11} A\exp(-\varepsilon A^{-1} I_1 t)C^\circ \sin(\alpha^{(1)} + \sigma)/\sin\theta_0$$

$$\theta_\varepsilon^v(t) = \theta_0 + \varepsilon I_1 D_{22}\sin\theta_0 k(\theta_0)t + \varepsilon D_{11} A\exp(-\varepsilon A^{-1} I_1 t)C^\circ \sin(\alpha^{(1)} - \mu)$$
$$\cos\sigma = \sin\mu = b^{(1)}\exp(\varepsilon A^{-1} I_1 t)/C^\circ$$

4. The case of a small constant torque. Let us consider the motion of a rigid body in the Lagrange case under the action of a torque which is constant in the body axes. Then the torques of the forces acting on the body have the form $M_i = \varepsilon^2 M_i{}^* = \varepsilon^2 M_i{}^\circ = \mathrm{const}$ $(i = 1, 2, 3)$. To construct an approximate solution of system (2.8) using the expressions for M_i, we apply the averaging procedure of Sect.2. The vector-valued function B_1 is determined as in (3.3), and the vector-valued function A_1 obtained from (2.13) has the following components:

$$A_1{}^{(1)} = -KD_{11}b\cos\theta - {}^1/_2K'D_{11}b\sin\theta \tag{4.1}$$
$$A_1{}^{(2)} = KD_{11}a\cos\theta + {}^1/_2K'D_{11}a\sin\theta$$
$$A_1{}^{(3)} = C^{-1}M_3{}^*,\ A_1{}^{(4)} = KD_{11},\ A_1{}^{(5)} = 0$$

The fourth and fifth components of u_1 are represented by (3.4). The function $A_2(x^*)$ is determined from (2.15) and may be written as

$$A_2{}^{(1)} = D_{12}b\,[\delta K\cos\theta - {}^1/_2AK^2D_{21}(3\cos^2\theta - 1) + {}^1/_2K'\delta\sin\theta + \tag{4.2}$$
$${}^1/_8AD_{21}(K'\sin\theta)^2 - {}^1/_2AK'D_{21}K\sin 2\theta]$$
$$A_2{}^{(2)} = -D_2a\,[\delta K\cos\theta - {}^1/_2AK^2D_{21}(3\cos^2\theta - 1) + {}^1/_2K'\delta\sin\theta +$$
$${}^1/_8AD_{21}(K'\sin\theta)^2 - {}^1/_2AK'D_{21}K\sin 2\theta]$$
$$A_2{}^{(3)} = 0,\ A_2{}^{(4)} = -KD_{12}\delta + AK^2D_{33}\cos\theta,\ A_2{}^{(5)} = 0$$

The solution of the averaged system of equations in the first approximation (2.16), where the coefficients are as in (4.1), is as follows for the slow and fast variables:

$$a^{(1)} = a^\circ\cos wt - b^\circ\sin wt,\ b^{(1)} = b^\circ\cos wt + a^\circ\sin wt \tag{4.3}$$
$$\delta^{(1)} = \varepsilon C^{-1}M_3{}^*t,\ \psi^{(1)} = \varepsilon K_0D_{11}t + \psi_0,\ \theta^{(1)} = \theta_0$$
$$\alpha^{(1)} = CA^{-1}r_0t - \varepsilon K_0D_{11}\cos\theta_0 t + {}^1/_2\varepsilon^2A^{-1}M_3{}^*t^2 + \varphi_0$$
$$\gamma^{(1)} = n_0t + {}^1/_2\varepsilon^2(C - A)C^{-1}A^{-1}M_3{}^*t^2$$

(the notation is the same as in (3.6)).

We note that the only component of the constant torque occurring in the solution of the first-approximation averaged system (4.3) is the component $M_3{}^*$ in the direction of the axis of symmetry. The projections $M_1{}^*$, $M_2{}^*$ of the perturbing torque vector cancel out on averaging. A comparison of formulae (4.3) for the slow variables $a^{(1)}$, $b^{(1)}$ with the parallel formulae (5.3) in /3/, with $K = \mathrm{const}$, shows that the formulae are identical.

By (2.26) and formulae (3.4), (4.2) and (4.3), the components of the function $x_\varepsilon{}^v(t)$ corresponding to the variables ψ and θ are

$$\psi_\varepsilon{}^v(t) = \psi_0 + \varepsilon K_0D_{11}t + V^{(1)} \tag{4.4}$$
$$V^{(1)} = \varepsilon^2tK_0{}^2D_{33}A\cos\theta_0 - {}^1/_2\varepsilon^3D_{22}M_3{}^*K_0t^2 - \varepsilon D_{11}AC^\circ\sin(\alpha^{(1)} + \varkappa)/\sin\theta_0$$
$$\theta_\varepsilon{}^v(t) = \theta_0 + \varepsilon D_{11}AC^\circ\sin(\alpha^{(1)} - \chi)$$
$$\cos\varkappa = \sin\chi = b^{(1)}/C^\circ$$

In the expression for $\theta_\varepsilon{}^v$, the bounded oscillating term involves the non-zero initial data a°, b°. The term $V^{(1)}$, as in the previous problem, corrects the formula $\omega_p = KC^{-1}r_0{}^{-1}$, already known from the approximate theory of gyroscopes, for the angular velocity of precession.

We note that the formulae for the angles of nutation and precession do not involve the parameters of the perturbing torques if attention is limited to the first approximation. The effect of the perturbations on the regular precession of the body is not taken into consideration in that case, so that construction of the second approximation is indispensable.

Going back to our example, when the restoring torque depends on the angle of nutation as in (2.7) and taking (2.13) into consideration, the solution of the first-approximation averaged system (2.16) for $a^{(1)}, b^{(1)}, \delta^{(1)}, \theta^{(1)}, \gamma^{(1)}$ has the form (4.3). Only the expressions for $\psi^{(1)}$ and $\alpha^{(1)}$ change, being written as follows:

$$\psi^{(1)} = D_{11}k(\theta_0)\, t + \psi_0 \tag{4.5}$$

$$\alpha^{(1)} = CA^{-1}r_0 t - D_{11}k(\theta_0)\cos\theta_0 t + {}^1/_2\varepsilon^2 A^{-1}M_3^* t^2 + \varphi_0$$

In (4.3)

$$w = D_{11}\{(mgl + \nu hz)\cos\theta_0 - {}^1/_2\nu hzs_0[2(h^2 + z^2)\cos\theta_0 - 5hz\cos^2\theta_0 + hz](h^2 + z^2 - 2hz\cos\theta_0)^{-3/2}\}$$

and $k(\theta_0)$ in (4.5) is given by formula (2.27) with $\theta = \theta_0$.

The components of the functions $x_\varepsilon^v(t)$ corresponding to the variables ψ, θ in our example are written as follows:

$$\psi_\varepsilon^v(t) = \psi_0 + D_{11}k(\theta_0)\, t + V^{(1)} \tag{4.6}$$

$$V^{(1)} = -{}^1/_2\varepsilon^2 D_{22}M_3^* k(\theta_0)\, t^2 + D_{33}A\cos\theta_0 k^2(\theta_0)\, t - \varepsilon D_{11}AC^\circ \sin(\alpha^{(1)} + \varkappa)/\sin\theta_0$$

$$\theta_\varepsilon^v(t) = \theta_0 + \varepsilon D_{11}AC^\circ \sin(\alpha^{(1)} - \chi)$$

$$\cos\varkappa = \sin\chi = b^{(1)}/C^\circ$$

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